Golfer's dilemma

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A ball rolling on a vertical cylinder exhibits a bizarre quasi-periodic motion. We explain this behavior in terms of Coriolis torque and study the phenomenon experimentally. The data confirm the predicted motion and the surprising invariance of the ratio between the horizontal and vertical frequencies. This result might account for the frustrating sight of a golf ball escaping from the hole an instant after it is putted in. © *2006 American Association of Physics Teachers.*
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I. INTRODUCTION

Roll a ball in a vertical cylinder with gravity downward. What does the motion look like? We assume that during the motion, the ball keeps in contact with the cylinder and does not "peel off," there is no dissipation, and the ball rolls and never slips. Of course, if the ball is started vertically, it will eventually roll straight down. So, suppose that the initial velocity has a nonzero horizontal component.

The majority of people whom we polled guessed that the motion would be either a spiral descent with increasing steepness or a spiral descent with shrinking pitch and asymptotic motion to a horizontal circle. In fact, the ball moves down *and up*; more precisely, the horizontal motion of the ball's center is a uniform revolution around the cylinder's axis and the vertical motion is a harmonic oscillation.

II. THEORY

This motion seems to have been pointed out first as a 19th-century Tripos (degree courses at Cambridge University) question. It is given in Routh[1] and solved by the Lagrangian method in Neĭmark and Fufaev.[2] Littlewood[3] mentions it as well ("Golfers are not so unlucky as they think"). This much, although somewhat forgotten, is in the classical literature. But, the strangeness of the motion does not stop here.

Given that there are horizontal and vertical oscillations, we may ask how they are coupled. For example, what is the ratio between the vertical and the horizontal periods? The answer turns out to be a universal constant that does not depend on any of the parameters in the problem—the radius r_b of the ball, the radius r_c of the cylinder, the mass m of the ball, the gravitational acceleration g, or the initial conditions. For a homogeneous ball, this ratio is $\sqrt{7/2}$.

We may also consider new variants of the problem. It is easy to check that if a ball of mass m rolls down a vertical plane, it travels along the parabola that would be traced by a sliding particle of mass $7m/5$. Therefore, the down-and-up motion is an effect of the curvature of the wall. If we now roll the ball on the outside of the cylinder (by imposing a constraint), we would expect an opposite effect, because in this case the wall curves away from, rather than toward the ball. Nevertheless, the ball's center executes a uniform revolution horizontally and a harmonic oscillation vertically, coupled again via $\sqrt{7/2}$. What if we spin the cylinder while the ball rolls? This spin drags the ball's point of contact with the cylinder, a non-Galilean transformation, so it should generate new dynamics. The outcome, however, is as before, and $\sqrt{7/2}$ is still there.

Classical solutions to the original problem rely on the fortunate accident that the governing equations are integrable. In this paper we show that the physical essence of the problem is the *Coriolis torque*; the latter yields a unified solution to all the variants of the problem and explains the appearance of a universal constant. Although the Coriolis torque acts on any spinning body in a rotating frame, we believe this example is one of the first in the literature where it is used to advantage.

A. Coriolis torque

What forces the ball up? Let $\boldsymbol{\omega}$ be the angular velocity of the ball about its center, and ω_z, ω_ϕ, ω_ρ its vertical, azimuthal, and radial components in cylindrical coordinates, respectively (see Fig. 1). While the ball is rolling down, $\omega_\phi < 0$ (for inside rolling). If we can find a mechanism that reverses the sign of ω_ϕ, then the ball will roll upward. The desired mechanism exists and it is the Coriolis torque.

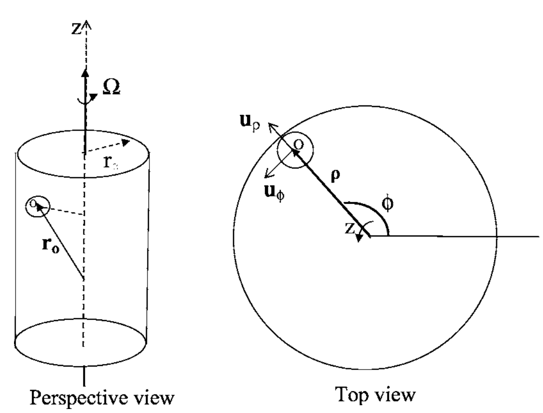

Fig. 1. The rotating frame. In the top view the unit vectors $\mathbf{u}_\phi$ and $\mathbf{u}_\rho$ correspond to ω_ϕ and ω_ρ.

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Take a frame rotating with angular velocity $\boldsymbol{\Omega}$ around the axis of the cylinder in such a way that the point of contact between the ball and the cylinder appears to move only vertically along a "contact line." In the rotating frame the ball feels the fictitious centrifugal, Euler, and Coriolis forces. The centrifugal force cancels the normal reaction of the cylinder. The Euler force $-m\dot{\boldsymbol{\Omega}}\times\mathbf{r}_0$ is present if $\boldsymbol{\Omega}$ varies in time; here, $\mathbf{r}_0$ denotes the position of the ball's center. Because $\dot{\mathbf{r}}_0$ is vertical, the net Coriolis force vanishes, $-2m\boldsymbol{\Omega}\times\dot{\mathbf{r}}_0=0$. But, the ball is subject to a torque which we call the Coriolis torque:

$$\boldsymbol{\tau}=\int \mathbf{r}\times[-2\boldsymbol{\Omega}\times(\boldsymbol{\omega}\times\mathbf{r})]dm, \tag{1}$$

where $\mathbf{r}$ is the radius vector measured from the ball's center. Equation (1) gives the total torque due to the Coriolis force acting at various points of the ball. To calculate $\boldsymbol{\tau}$, recall that for any body with angular velocity $\boldsymbol{\omega}_0$, the moment of inertia $\mathbf{I}$ is the matrix such that

$$\int \mathbf{r}\times(\boldsymbol{\omega}_0\times\mathbf{r})dm=\mathbf{I}\boldsymbol{\omega}_0. \tag{2}$$

The integral is the total angular momentum of the body; it is linear in $\boldsymbol{\omega}_0$ and so can be represented by a matrix $\mathbf{I}$. If the density distribution is spherically symmetric, $\mathbf{I}$ reduces to the scalar I

$$I=\frac{2}{3}\int dmr^2. \tag{3}$$

(A factor of $2/3$ is present because $\mathbf{r}$ is measured from the center, not from an axis.) By the vector identity $\mathbf{a}\times[\mathbf{b}\times(\mathbf{c}\times\mathbf{a})]=\mathbf{a}^2\mathbf{c}\times\mathbf{b}-[\mathbf{a}\times(\mathbf{c}\times\mathbf{a})]\times\mathbf{b}+\mathbf{a}\times[(\mathbf{b}\times\mathbf{c})\times\mathbf{a}]$, we obtain

$$\boldsymbol{\tau}=-2\int dm\{r^2\boldsymbol{\omega}\times\boldsymbol{\Omega}-[\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r})]\times\boldsymbol{\Omega}+\mathbf{r}\times[(\boldsymbol{\omega}\times\boldsymbol{\omega})\times\mathbf{r}]\} \tag{4a}$$

$$=-2\left(\frac{3}{2}I(\boldsymbol{\omega}\times\boldsymbol{\Omega})-(I\boldsymbol{\omega})\times\boldsymbol{\Omega}+I(\boldsymbol{\Omega}\times\boldsymbol{\omega})\right) \tag{4b}$$

$$=I\boldsymbol{\omega}\times\boldsymbol{\Omega}. \tag{4c}$$

The horizontal uniform revolution is readily understood. The only vertical torque that influences $\dot{\boldsymbol{\Omega}}$ is produced by the horizontal forces: the Euler force on the ball's center and the longitudinal reaction from the cylinder acting at the point of contact. The latter has no effect if the torques are calculated about the contact point. The torque $\boldsymbol{\tau}_E$ due to the Euler force $-m\dot{\boldsymbol{\Omega}}\times\mathbf{r}_0$ is

$$\boldsymbol{\tau}_E=-r_b\mathbf{u}_\rho\times(-m\dot{\boldsymbol{\Omega}}\times\mathbf{r}_0)=mr_br_c\dot{\boldsymbol{\Omega}}. \tag{5}$$

Alternately, because the ball is rolling without slipping, $r_c\Omega=r_b\omega_z$, and so by the parallel axes theorem and the relation $\boldsymbol{\tau}_E=\dot{\mathbf{L}}$, where $\mathbf{L}$ is the angular momentum about the contact point, we have

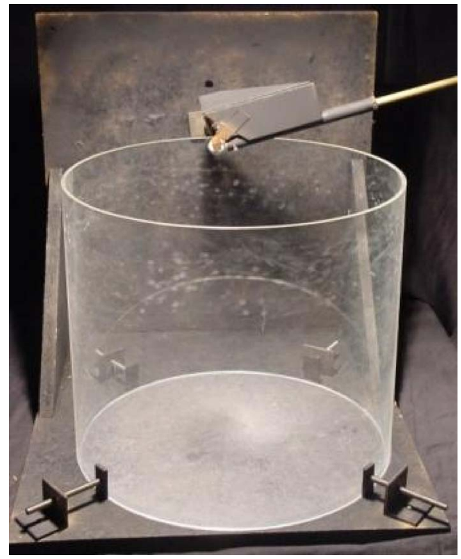

Fig. 2. Cylinder and spring gun.

$$\boldsymbol{\tau}_E=(I+mr_b^2)\frac{r_c}{r_b}\dot{\boldsymbol{\Omega}}. \tag{6}$$

The only way that Eqs. (5) and (6) can be simultaneously satisfied is for $\dot{\boldsymbol{\Omega}}$ to be 0, so $\boldsymbol{\Omega}$ is constant and the horizontal motion is a uniform revolution.

To understand the vertical harmonic oscillation, it is natural to consider the angular momentum $\mathbf{L}$ not about the ball's center, but about a fixed reference point on the contact line. Let $I=mr_b^2K$; K is a dimensionless number that reflects only the geometry of the density distribution. By the parallel axes theorem, we have

$$\mathbf{L}=\begin{pmatrix} I & 0 & 0 \\ 0 & I+mr_b^2 & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix}\omega_z \\ \omega_\phi \\ \omega_\rho\end{pmatrix}, \tag{7}$$

as if the ball were a free oblate spheroid. We invert the matrix in Eq. (7) to obtain an expression of $\boldsymbol{\omega}$ versus $\mathbf{L}$. If we use Eq. (4c), we have for the Coriolis torque

$$\boldsymbol{\tau}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{K}{1+K} & 0 \\ 0 & 0 & 1 \end{pmatrix}(\mathbf{L}\times\boldsymbol{\Omega}). \tag{8}$$

The torque $\boldsymbol{\tau}$ causes $\mathbf{L}$ to precess elliptically around the vertical axis. The additional constant torque $\boldsymbol{\tau}_g$ due to gravity merely shifts the equilibrium position for the precession.

If we write $\dot{\mathbf{L}}=\boldsymbol{\tau}+\boldsymbol{\tau}_g$, we obtain

$$\dot{\omega}_z=0, \tag{9a}$$

$$\dot{\omega}_\phi=\frac{K}{1+K}\Omega\omega_\rho\mp\frac{g}{(1+K)r_b}, \tag{9b}$$

$$\dot{\omega}_\rho=-\Omega\omega_\phi. \tag{9c}$$

Equation (9) shows that the horizontal and vertical motion is decoupled. The upper sign corresponds to inside rolling, the lower sign to outside rolling. No slip means that the ball and the cylinder have zero relative velocity at their point of contact. The height z of the ball's center must satisfy

$$\dot{z} = \pm r_b \omega_\phi. \tag{10}$$

From Eqs. (9b), (9c), and (10), it follows that

$$\ddot{z} + \frac{\Omega^2}{1+1/K} z + \frac{g}{1+K} = 0. \tag{11}$$

That is, z oscillates harmonically with angular frequency $\Omega_V = \Omega/\sqrt{1+1/K}$ (as do ω_ϕ and ω_ρ).

The Coriolis torque $\boldsymbol{\tau}$ also explains the appearance of $\sqrt{7/2}$. Whether the ball is rolling on the inside or outside of the cylinder does not affect $\boldsymbol{\tau}$. Neither do r_b, r_c, nor the gravitational acceleration g. However, the magnitude of $\boldsymbol{\tau}$ is proportional to Ω, and so the rate of precession of $\mathbf{L}$ is proportional to Ω. This proportionality explains that the ratio Ω/Ω_V of the vertical to horizontal periods is a universal constant $\sqrt{1+1/K}$, which is $\sqrt{7/2}$ for a homogeneous ball (and $\sqrt{5/2}$ for a spherical shell).

Note that if the velocity of the ball's center is initially horizontal, then the amplitude A of the vertical oscillation is

$$A = \left| \frac{g}{K\Omega^2} \pm \frac{r_b \omega_\rho(0)}{\Omega} \right|. \tag{12}$$

In particular, $A=0$ when $\omega_\rho(0) = \mp g/K r_b \Omega$; with this initial condition, the ball rolls along a perfect horizontal circle.

B. Rolling constraint

The key assumption in the theory we have presented is the rolling, or no-slip constraint, whose applicability we now examine for vertical motion and inside rolling (the outside case is similar). Let μ be the coefficient of static friction between the ball and the cylinder. The normal reaction of the cylinder balances the centrifugal force and has magnitude $m(r_c - r_b)\Omega^2$. The tangential reaction has magnitude $mg - mz\Omega^2/(1+1/K)$, measuring z from the midpoint of its oscillation. Slipping is avoided as long as

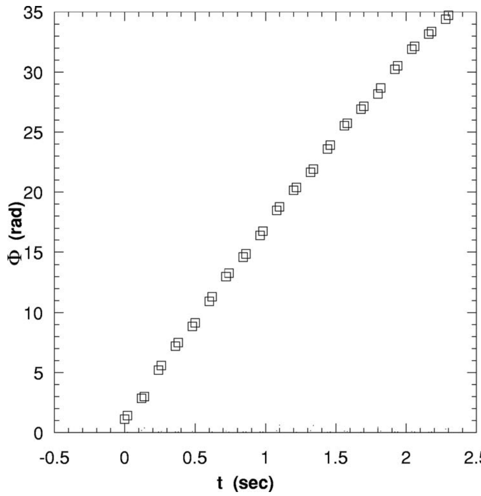

Fig. 4. Plot of $\phi(t)$ for the trajectory of Fig. 3.

$$g + \frac{A\Omega^2}{1+1/K} < \mu(r_c - r_b)\Omega^2. \tag{13}$$

Thus, provided Ω is chosen to be large enough, the motion we describe is physically possible.

C. Open problems

The rolling requirement constrains not only the position or orientation of the ball, but also its velocity. This requirement is an example of a *nonholonomic* mechanical system. In geometric language, we are not restricted to the tangent bundle of a submanifold of the configuration space, but rather to a subbundle of the tangent bundle of the configuration space. It would be interesting to find other nonholonomic systems that involve a universal constant analogous to $\sqrt{7/2}$ and to clarify how such a constant relates to the nature of the subbundle. Perhaps an analysis along the lines of the recent work of Ref. 4 would be helpful.

In our problem, every motion (except straight descents) is quasi-periodic, a behavior typical of a completely integrable

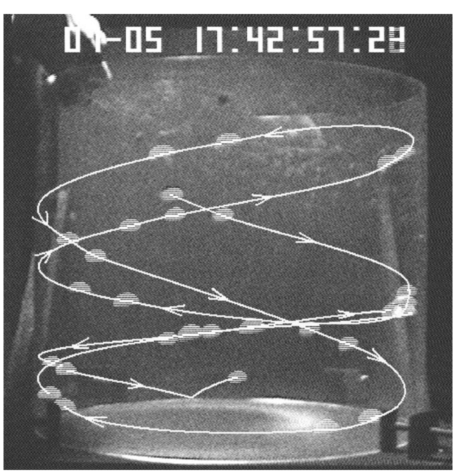

Fig. 3. The trajectory obtained by superimposing successive snapshots of the ball thrown by hand. We see two oscillations and then a bounce on the floor.

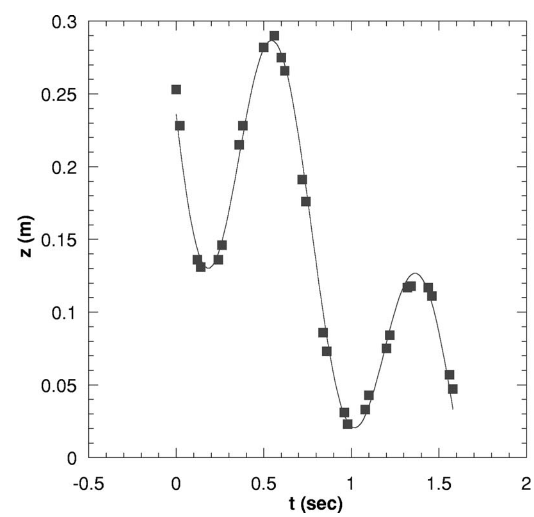

Fig. 5. Plot of $z(t)$ for the trajectory of Fig. 3.

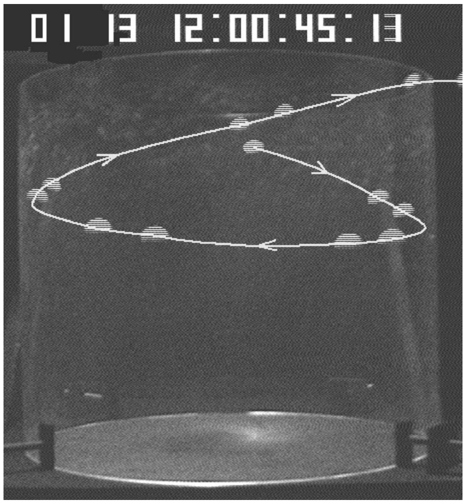

Fig. 6. Trajectory of a throw with high initial speed, leading to an exit of the ball.

system. At present there is no satisfactory theory of nonholonomic integrability. The fact that the ratio of frequencies is locked at $\sqrt{7/2}$ suggests that the system is fully "degenerate," somewhat like a multi-dimensional linear spring in holonomic theory. Perturbation theory for integrable systems (for example, *à la* Kolmogorov–Arnol'd-Moser) is notoriously difficult in the degenerate case.

III. EXPERIMENTAL SETUP

In our experiment a computer mouse ball of diameter 24.15 mm with a metallic core and latex surface was launched by hand or with a spring gun at an angle α to the horizontal and initial speed V_0, along the interior of a transparent Altuglas cylinder of internal diameter 0.39 m and height 0.35 m (see Fig. 2). We estimate the friction coefficient to be $\mu=1/\sqrt{3}$. Hence, the critical angular speed for rolling (see Sec. II B) is 9.7 rad/s. The trajectory of the ball was photographed by an analog camcorder (JVC GR-M72S) with adjustable shutter speed, whose signal was fed to the image capture card of a personal computer able to handle 12 frames/s. The environment outside the cylinder was painted black to avoid reflections.

The rolling constraint was achieved by choosing an angular frequency $\Omega=d\phi/dt>16$ rad/s, that is, at least 2.5 turns/s and an initial horizontal speed $V_0 \cos\alpha$ of 2 m/s. Because the speed of the ball was of the order of 3 m/s, the shutter speed was set to 1/4000 s; the streak of the ball image was therefore about 0.75 mm, negligible in comparison to the ball diameter. A halogen spotlight provided uniform lighting with sufficient brightness for the short exposure times used. Because the camcorder operated in interlaced mode, the even and odd lines gave two different views of the ball at times separated by 0.02 s; we thus obtained two delayed positions of the ball per image. A timer stamped each image with a precision of 0.01 s.

One hundred images of each run were captured, and the first 10 to 20 of them retained; after this time the ball hit the wooden floor. The images were processed with SCIONIMAGE software to yield the position of the ball in cylindrical coordinates $z(t), \phi(t)$ at time t. For each run, the origin $t=0$ is at the first recorded ball position after the throw.

Table I. The ratio $R=\Omega/\Omega_V$ for 12 throws, four of them by a spring gun (SG) and eight by hand throw (HT), at different launch angles α and initial speeds V_0.

Launch	α (rad)	V_0 (m/s)	R
SG1	0.339	3.382	2.224
SG2	0.427	3.502	2.159
SG4	0.400	3.090	1.920
SG5	0.424	3.114	2.048
HTE2	0.658	3.021	2.224
HTE3	0.423	3.104	2.095
HTE4	0.482	3.114	2.048
HTE51	0.411	3.151	2.011
HTE52	0.398	3.016	2.011
HTL1	0.667	2.673	2.011
HTL2[a]	0.564	3.210	2.117
HTB3	0.586	3.112	2.035
$\bar{R}$			2.069
Standard deviation			0.100

[a]This throw resulted in the ball popping out of the cylinder, as can be seen in Fig. 6.

IV. EXPERIMENTAL RESULTS

The recorded trajectories clearly show the vertical oscillations; see Fig. 3 for a trajectory showing two oscillations obtained by a hand launch. Figure 4 shows the graph of $\phi(t)$ with $\Omega=d\phi/dt$ nearly constant. In this case a linear fit gives $\Omega=15.8\pm0.5$ rad/s. Figure 5 shows the graph of $z(t)$. The mean altitude of the ball drifts down as expected from dissipation, so we use the form $z(t)=Ae^{-at}\cos(\Omega_V t)+bt+c$, where Ω_V is the angular frequency of the vertical oscillation; Ω_V given by this fit is 7.66 ± 0.03 rad/s, so the ratio $R=\Omega/\Omega_V$ is 2.048, slightly greater than the theoretical prediction. The launch angle α and the initial speed V_0 of each throw were deduced from the first recorded position of the ball after launching. Several throws were made with different α and V_0. Table I shows the angular frequency ratios R and their mean value $\bar{R}$.

For a latex ball with internal metallic core radius R_{in} and external radius R_{ext}, the moment of inertia is $I=2\pi\frac{4}{3}(\rho_{\mathrm{in}}R_{\mathrm{in}}^5/5+\rho_{\mathrm{ext}}(R_{\mathrm{ext}}^5-R_{\mathrm{ext}}^5)/5)$, where ρ_{in} and ρ_{ext} are the densities of metal and latex. These length and mass measurements of the ball give the theoretical value of $R=\sqrt{1+1/K}=2.009\pm0.1$, in reasonable agreement with $\bar{R}=2.069$ with a standard deviation of 0.1.

V. CONCLUSION

We have analyzed and measured the bizarre vertical oscillations of a ball rolling in a cylinder and have verified that the observed ratio Ω/Ω_V is constant and agrees with the theoretically predicted value $\sqrt{1+1/K}$. We are pleased to have reproduced in a laboratory setting the cunning escape of a golf ball from a hole into which it has fallen.

ACKNOWLEDGMENTS

Several years ago Hugh Hunt and Mark Warner of Cambridge University did a very nice informal experiment of this motion. T.T. thanks them for inspiring discussions. We thank the skilled technical support of Jérôme Jovet and Alain Roger. The undergraduate students (L.A.G, B.C., and E.R.) performed the experimental part of this work in fulfillment of the Physique Expérimentale education unit of Université Paris 7.

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[1]E. J. Routh, *Dynamics of a System of Rigid Bodies. Advanced Part* (Dover, New York, 1955) (Reprinted from the 6th ed., Macmillan, New York, 1905), Chap. V, Art. 225–227.

[2]Ju. I. Neĭmark and N. A. Fufaev, *Dynamics of Nonholonomic Systems* (American Mathematical Society, 1972), Chap. III, Sec. 2, Exam. 2.

[3]J. E. Littlewood, *Miscellany* (Cambridge University Press, Cambridge, 1986), p. 46.

[4]C.-M. Marle, "On symmetries and constants of motion in Hamiltonian systems with nonholonomic constraints," in *Classical and Quantum Integrability* (Banach Center Publications, Warsaw, 2003), Vol. 59, pp. 223–242.

A dynamical top with nine adjustments, used by Maxwell to illustrate the classical laws of rotation. He wrote, "In this communication, I intend to confine myself to that part of the subject which the top is intended to illustrate, namely, the alteration of the axis in a body rotating freely around its centre of gravity... . The mathematical difficulties of the theory of rotation arise chiefly from the want of geometrical illustrations and sensible images, by which we might fix the results of analysis in our minds." (Photo by Jay M. Pasachoff, Williams College, published with permission of the Cavendish Laboratory, Cambridge, and photographed with the assistance of Gordon Squires.)